JOURNAL OF APPROXIMATION THEORY 69, 173–187 (1992)

Strictly and Uniformly Monotone Musielak–Orlicz Spaces and Applications to Best Approximation

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Communicated by Allan Pinkus

Received June 21, 1990; revised April 30, 1991

In this paper several monotonicity properties of the Luxemburg norm in Musielak–Orlicz spaces $L_\phi(\mu)$ and $E_\phi(\mu)$ over nonatomic measure spaces are characterized in terms of the function ϕ. For $L_\phi(\mu)$ it is proved that all these properties coincide with the absolute continuity of the norm and $\phi > 0$. Some applications to best approximation are given, even for general Banach lattices.

1. PRELIMINARIES

Let X be a Banach lattice with a lattice norm $\|\cdot\|$. Following [2, Chap. XV], the norm $\|\cdot\|$ is said to be uniformly monotone (UM), if for every $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that $\|f+g\| \geqslant 1+\eta(\varepsilon)$ whenever $f, g \in X^+$ (positive cone in X), $\|f\| = 1$, and $\|g\| \geqslant \varepsilon$. We will call such a space a UM space ("UMB" space in [2]). We say that the norm is strictly monotone (STM) if $\|f+g\| > \|f\|$ for all nonnegative $f, g \in X$ with $\|g\| > 0$. In this case we call X an STM space. We will also say that X has the UM or STM property, respectively. An equivalent condition for X to be an STM space is that $\|f-g\| < \|f\|$ whenever $f \geqslant g \geqslant 0$ and $g \neq 0$.

PROPOSITION 1.1. *The following statements are equivalent.*

(a) *X is a UM space.*

(b) *For every $\varepsilon > 0$ there exists a $\delta(\varepsilon) \in (0, 1)$ such that $\|f-g\| < 1-\delta(\varepsilon)$ whenever $f \geqslant g \geqslant 0$, $\|f\| = 1$, and $\|g\| > \varepsilon$.*

(c) *For all nonnegative sequences (f_n), (g_n) in X satisfying $\|f_n\| = 1$ and $\|f_n + g_n\| \to 1$ there holds $\|g_n\| \to 0$.*

(d) *For all sequences (f_n), (g_n) in X satisfying $\|f_n\| = 1$, $f_n \geqslant g_n \geqslant 0$, and $\|f_n - g_n\| \to 1$ there holds $\|g_n\| \to 0$.*

The conditions (a) and (b) can be expressed in terms of a modulus of uniform monotonicity. Namely X is a UM space precisely when $\eta(\varepsilon) =$

0021-9045/92 $3.00

$\inf\{\|f+g\|-1: f, g\geqslant 0,\ \|f\|=1,\ \|g\|\geqslant\varepsilon\}>0$ or, equivalently, $\delta(\varepsilon)=\inf\{1-\|f-g\|: f\geqslant g\geqslant 0,\ \|f\|=1,\ \|g\|\geqslant\varepsilon\}>0$, where $\varepsilon\in(0,1)$. It can be verified that for such ε the following inequalities hold true:

$$\frac{\delta(\varepsilon)}{1-\delta(\varepsilon)}\geqslant\eta(\varepsilon)\geqslant\delta\left(\frac{\varepsilon}{1+\varepsilon}\right)\Big/\left(1-\delta\left(\frac{\varepsilon}{1+\varepsilon}\right)\right).$$

Clearly, each UM space is an STM space. In UM spaces the norm is order continuous ($X\in(A)$; $0\leqslant x_\alpha\downarrow 0$ implies $\|x_\alpha\|_\phi\to 0$) and monotonically complete ($X\in(B)$; $0\leqslant x_\alpha\downarrow$ and $\sup_\alpha\|x_\alpha\|<+\infty$ imply that $\sup_\alpha x_\alpha\in X$), [2, Chap. XV, Theorem 22]. In other words X is a KB space [12, Chap. X, Sect. 4.4]. Let us point out that X is a KB space if and only if it is order continuous and monotonically complete for sequences only.

For example, L_p-spaces with $1\leqslant p<+\infty$ are UM spaces, but the space L_∞ is not even an STM space. The problem is how to distinguish STM or UM Musielak–Orlicz spaces.

The former results concerning the characterization of the STM and the UM property in Orlicz spaces can be found in [1, Theorem 4.4; 4, Theorem 4]; see also [5, Theorem 34]. In [1, 4] some other monotonicity properties are characterized. In [4, 5] the case of the Orlicz norm is also considered.

The first aim of this paper is to show that a Musielak–Orlicz space over nonatomic measure space endowed with the Luxemburg norm is either a UM and hence an STM space, or it is not even an STM space. In the case of purely atomic measure spaces there exist STM Musielak–Orlicz spaces which are not UM spaces [19]. Our approach is via characterizations of UM and STM Musielak–Orlicz spaces $L_\phi(\mu)$ in terms of the modular $I_\phi(\cdot)$ and then in terms of the function ϕ only. Hence it follows, in the nonatomic case, that Musielak–Orlicz spaces are UM spaces (equivalently: STM spaces) precisely when $\phi>0$ and they have order continuous (lattice) norm. Our second aim is to apply these results to some best approximation problems.

The UM and STM properties can be viewed as the uniform rotundity (UR) and the rotundity (R) restricted to the positive cone X^+ in X, respectively. In analogy to intermediate rotundity properties for UR and R (WUR, CWUR, LUR, HR; cf. [6, 23, 14]) some intermediate properties for UM and STM are defined below. The relation between the UM and UR property as well as between the STM and R property follows immediately from Propositions 1.2, 1.3, and the definitions above.

PROPOSITION 1.2. *The following statements are equivalent.*

(a) *X is a UR space, i.e., for each $\varepsilon>0$ there exists a $\delta(\varepsilon)\in(0,1)$ such that $\|(f+g)/2\|<1-\delta(\varepsilon)$ whenever $\|f\|$, $\|g\|\leqslant 1$ and $\|f-g\|\geqslant\varepsilon$.*

(b) *For every* $\varepsilon>0$ *there exists an* $\eta(\varepsilon)\in(0,1)$ *such that* $\|f\|=1$ *and* $\|g\|\geqslant\varepsilon$ *imply* $\max_{\mp}\|f\mp g\|>1+\eta(\varepsilon)$.

Proof. (a) ⇒ (b). If (b) is not satisfied then there exists an $\varepsilon_1>0$ such that for every $\eta\in(0,1)$ one can find $f, g\in X$ satisfying $\|f\|=1$, $\|g\|>\varepsilon_1$ with $\max_{\mp}\|f\mp g\|\leqslant 1+\eta$. Let $f_{\mp}=(f\mp g)/(1+\eta)$. We have $\|f_{\mp}\|\leqslant 1$, $\|f_- - f_+\|\geqslant 2\varepsilon_1/(1+\eta)>\varepsilon_1$. With this ε_1 and $\eta=\delta(\varepsilon_1)>0$ from (a) it follows that $\|(f_- + f_+)/2\|=1/(1+\delta(\varepsilon_1))>1-\delta(\varepsilon_1)$, a contradiction.

(b) ⇒ (a). If not, without loss of generality, there exist $\varepsilon_1>0$ and sequences (f_n), (g_n) of norm-one functions such that $\|f_n-g_n\|\geqslant\varepsilon_1$ and $\alpha_n=\|f_n+g_n\|/2>1-1/n$. Let $h_n=(f_n+g_n)/(2\alpha_n)$. Then $\|h_n\|=1$ and $\alpha_n\to 1$. For $z_n=(f_n-g_n)/(2\alpha_n)$ we have $\|z_n\|\geqslant\varepsilon_1/2$. Applying (b) with $\varepsilon=\varepsilon_1/2$ there exists $\delta(\varepsilon)>0$ such that for h_n and z_n given above, we have $\max_{\mp}\|h_n\mp z_n\|>1+\delta(\varepsilon)$. On the other hand $\|h_n-z_n\|=1/\alpha_n\to 1$ and this contradiction finishes the proof.

Remark. Denoting the modulus of uniform monotonicity by $\eta_+(\varepsilon)$ and the modulus of uniform rotundity resulting from (b) by $\eta(\varepsilon)$ it follows that $0\leqslant\eta(\varepsilon)\leqslant\eta_+(\varepsilon)\leqslant\varepsilon$. For $L_1(\mu)$ we have $\eta(\varepsilon)=0$ but $\eta_+(\varepsilon)=\varepsilon$.

PROPOSITION 1.3. *The following conditions are equivalent.*

(a) *X is rotund (R), i.e.,* $\|f+g\|=2$, $\|f\|$, $\|g\|\leqslant 1$ *imply* $f=g$.

(b) *For each nonzero f,* $\max_{\mp}\|f\mp h\|>\|f\|$ *whenever* $\|h\|>0$.

The simple proof is omitted. Collecting all these facts it follows that UR ⇒ UM (cf. [2, Example 4, Chap. XV, 14]), UM ⇒ STM, UR ⇒ R ⇒ STM.

We call a Banach lattice X locally uniformly monotone (LUM) if $f\geqslant g_n\geqslant 0$ $(n\in\mathbb{N})$ with $\|f\|=1$, $\|f-g_n\|\to 1$ imply $Ug_n\|\to 0$. X is said to be weakly uniformly monotone (WUM), if for each positive functional $l\in X^*$ (the Banach dual to X) and all sequences $f_n\geqslant g_n\geqslant 0$ with $\|f_n\|=1$ the condition $\|f_n-g_n\|\to 1$ implies $l(g_n)\to 0$. We call X weakly uniformly monotone in the second sense (CWUM) (cf. the CWUR property in [14]), if for $f_n\geqslant g_n\geqslant 0$ with $\|f_n\|=1$ and positive functionals $l\in S(X^*)$ (the unit sphere in X^*) the condition $l(f_n-g_n)\to 1$ implies that $\|g_n\|\to 0$. Localizations of these two properties (we consider constant sequences $f_n=f$) lead to the concepts of WLUM and CWLUM spaces, respectively (see [6, 23, 14] for the respective rotundity properties and further references). Finally, we say that a Banach lattice X possesses the H^+ property (an analogy to the Radon–Riesz property H; e.g., [23, 14]), if $f\geqslant f_n\geqslant 0$, $\|f\|=1$, and the weak convergence $f_n \to f$ imply the convergence of $\|f-f_n\|$ to zero. Since X is a Banach lattice, in these definitions one can deal equivalently with the functionals l from the whole space X^* or $S(X^*)$, respectively.

The following implications are evident: UM $\Rightarrow$ LUM $\Rightarrow$ WLUM, LUM $\Rightarrow$ CWLUM $\Rightarrow H^+$ and STM (i.e., H^+STM), UM $\Rightarrow$ WUM $\Rightarrow$ WLUM $\Rightarrow$ STM, UM $\Rightarrow$ CWUM $\Rightarrow$ CWLUM.

We leave the study of these properties for Banach lattices to another occasion. Let us point out only that the H^+ property of Banach lattices always yields the order continuity in X. Moreover, a Banach lattice X is a CWLUM space precisely when it is an STM space with an order continuous norm.

Let (T, Σ, μ) be a σ-finite, complete (non-trivial), positive measure space and $\phi(r, t)$: $R_+ \times T \to \bar{R}_+$ be a function such that for μ-a.e. $t \in T$, $\phi(0, t)=0$, $\phi(\cdot, t)$ is non-trivial (continuous at zero with nonzero values), convex, and lsc. Moreover let $\phi(r, \cdot)$ be measurable, for all $r>0$. Musielak–Orlicz spaces $L_\phi(\mu)$ [21, 15, 16, 7] consist of all μ-measurable functions f: $T \to \bar{R}$ such that $I_\phi(\alpha f)=\int_T \phi(\alpha\,|f(t)|, t)\,d\mu < +\infty$ for some $\alpha>0$ (depending on f). When endowed with the Luxemburg norm $\|\cdot\|_\phi$ it becomes a Banach lattice under the natural ordering (e.g., [24, 21]), where $\|f\|_\phi=\inf\{\lambda>0$: $I_\phi(f/\lambda)\leqslant 1\}$. The function ϕ is said to satisfy a Δ_2 condition ($\phi \in \Delta_2$) if there exist a set T_0 of zero measure, a constant $K>0$, and an integrable (non-negative) function h, such that for all $t \in T\backslash T_0$ and $r>0$ there holds $\phi(2r, t) \leqslant K\phi(r, t)+h(t)$.

In the following we will write, for short, $\phi>0$ or $\phi<+\infty$, if for μ-a.e. $t \in T$ the function $\phi(\cdot, t)$ is strictly positive (except zero) or assumes finite values only, respectively. Let $L^a_\phi(\mu)=\{f \in L_\phi(\mu)$: $|f| \geqslant f_n \downarrow 0 \Rightarrow \|f_n\|_\phi \downarrow 0\}$ be a subspace of functions with order continuous norm and $E_\phi(\mu)=\{f \in L_\phi(\mu)$: $I_\phi(\alpha f)<\infty$ for all $\alpha>0\}$. Then $E_\phi(\mu) \subset L^a_\phi(\mu) \subset L_\phi(\mu)$ as closed ideals (see [24, p. 17; 12; 25]). If $\phi<+\infty$ then $E_\phi(\mu)$ is super order dense in $L_\phi(\mu)$ and $L^a_\phi(\mu)=E_\phi(\mu)$ [24, p. 19]. $L_\phi(\mu)$ has an order continuous norm precisely when $L_\phi(\mu)=L^a_\phi(\mu)$. Clearly the norm in $E_\phi(\mu)$ is order continuous. Since ϕ is continuous at zero, $L_\phi(\mu)$ is decomposable (cf. [7] for bibliography). Also, if $\phi<+\infty$ then $E_\phi(\mu)$ is decomposable as well. In particular this means that there exists an increasing family $\{T_n\} \subset \Sigma$ of sets of finite measure with $\bigcup_n T_n \overset{\mu}{=} T$ such that $1_{T_n} \in L_\phi(\mu)$ or $1_{T_n} \in E_\phi(\mu)$, respectively, for all $n \in \mathbb{N}$ (see [12, Chap. IV, Sect. 3]).

We call the modular $I_\phi(\cdot)$ ϕ-uniformly monotone (uniformly monotone) if for each $\varepsilon>0$ there exists an $\eta(\varepsilon)>0$ such that $f \geqslant g \geqslant 0$ in $L_\phi(\mu)$ with $I_\phi(f)=1$ and $I_\phi(g)\geqslant\varepsilon$ (resp. with $\|f\|_\phi=1$ and $\|g\|_\phi \geqslant \varepsilon$) imply that $I_\phi(f-g)\leqslant 1-\eta(\varepsilon)$. Also $I_\phi(\cdot)$ is said to be ϕ-strictly monotone (strictly monotone), if $f \geqslant g \geqslant 0$ in $L_\phi(\mu)$ with $I_\phi(f)=1$ and $I_\phi(g)>0$ (resp. $\|f\|_\phi=1$ and $\|g\|_\phi>0$) imply $I_\phi(f-g)<1$. We will shortly write that $I_\phi(\cdot)$ is ϕ-UM, UM, ϕ-STM, and STM modular, respectively. Clearly each

UM modular is ϕ-UM and therefore ϕ-STM. Also each STM modular $I_\phi(\cdot)$ is ϕ-STM. Moreover, we have the following proposition.

PROPOSITION 1.4. *The modular $I_\phi(\cdot)$ is always ϕ-UM. Consequently the notions of ϕ-UM and ϕ-STM modulars coincide (μ-arbitrary).*

Proof. Let $\varepsilon > 0$ be arbitrary. To prove the first part of the proposition we shall prove that $\delta(\varepsilon) = (\varepsilon/2)^2$ is a good choice. Let $f \geqslant g \geqslant 0$ be such that $I_\phi(f) = 1$ and $I_\phi(g) \geqslant \varepsilon$. Let $A = \{t \in T: g(t) \leqslant \varepsilon f(t)/2\}$. Then $\varepsilon \leqslant I_\phi(g) = I_\phi(g 1_A) + I_\phi(g 1_{T \setminus A})$ imply $I_\phi(g 1_{T \setminus A}) \geqslant \varepsilon/2$. Next, $I_\phi(f) - I_\phi(f-g) \geqslant I_\phi(f 1_{T \setminus A}) - I_\phi((f-g)\, 1_{T \setminus A}) \geqslant I_\phi(f 1_{T \setminus A}) - (1-\varepsilon/2)\, I_\phi(f 1_{T \setminus A}) \geqslant (\varepsilon/2) \times I_\phi(g 1_{T \setminus A}) \geqslant (\varepsilon/2)^2 = \delta(\varepsilon)$, since $f(t) - g(t) < (1-\varepsilon/2)\, f(t)$ on $T \setminus A$, and the first assertion is proved. The second one follows immediately since each ϕ-UM modular is ϕ-STM.

PROPOSITION 1.5. *If $L_\phi(\mu)$ is an STM space then $\phi > 0$. The same is true for $E_\phi(\mu)$ whenever $\phi < +\infty$.*

Proof. We proceed by a contradiction. Let $A \subset \{t \in T: \exists (r > 0)\ \phi(r, t) = 0\}$, $\mu(A) > 0$, and $B = T \setminus A$. Since T is not an atom we can assume that $\mu(B) > 0$. There exists a measurable selector $g(t) \in \Gamma(t) = \{r > 0: \phi(r, t) = 0\}$, where $t \in A$. Since $L_\phi(\mu)$ is decomposable without loss of generality we can assume that $g 1_A \in L_\phi(\mu)$. Let $h \in L_\phi(\mu)$ be such that $\|h 1_B\|_\phi = 1$ and $h \geqslant 0$. Define $f = h 1_B + g 1_A$. Clearly $f \geqslant g 1_A \geqslant 0$, $g 1_A \neq 0$, and $1 = \|h 1_B\|_\phi \leqslant \|f\|_\phi$, $I_\phi(f) = I_\phi(f - g 1_A) = I_\phi(h 1_B) \leqslant 1$. Consequently $1 = \|f\|_\phi = \|f - g 1_A\|_\phi\ (= \|h 1_B\|_\phi)$, a contradiction. For the space $E_\phi(\mu)$ we proceed analogously. In this case we apply the decomposability of $E_\phi(\mu)$.

Remark. It can be proved that the STM property of the $L_\Phi(\mu)$ space implies that $\phi < +\infty$. This, however, will be implicitly contained in $\phi \in \Delta_2$.

2. STRICTLY AND UNIFORMLY MONOTONE MUSIELAK–ORLICZ SPACES

Recall some basic facts concerning relations between the modular $I_\phi(\cdot)$ and the Luxemburg norm $\|\cdot\|_\phi$ in the Musielak–Orlicz space $L_\phi(\mu)$ and in the space $E_\phi(\mu)$. Everywhere below μ is assumed to be nonatomic (the purely atomic case is considered in [19]).

PROPOSITION 2.1 (Hudzik [10]). *The following statements are equivalent.*

(a) *ϕ satisfies the Δ_2 condition.*

(b) *$\|f\|_\phi = 1$ implies $I_\phi(f) = 1$.*

(c) $\|f_n\|_\phi \nearrow 1$ *implies* $I_\phi(f_n) \to 1$.

(d) $L_\phi(\mu)$ *does not contain an isometric copy of* l_∞.

(e) $L_\phi(\mu)$ *does not contain a lattice isometric copy of* l_∞.

(f) $L_\phi(\mu) = L_\phi^a(\mu)$, *i.e., the norm* $\|\cdot\|_\phi$ *is order continuous on* $L_\phi(\mu)$.

Remarks. The proofs of the equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (d) can be found in [9] (the assumption on the extended continuity of the function $\phi(\cdot, t)$ in [9] can be dropped); see also [7] where μ is assumed separable. From [12, Chap. X, Sect. 4] it follows that (f) $\Rightarrow$ (e). Also (a) $\Rightarrow$ (f) since $E_\phi(\mu) \subset L_\phi^a(\mu) \subset L_\phi(\mu)$ and (a) implies that $E_\phi(\mu) = L_\phi(\mu)$ [15]. The implication (c) $\Rightarrow$ (b) is trivial and (b) $\Rightarrow$ (c) follows from [9, Lemma 1.5]. Next, (d) $\Rightarrow$ (e) is trivial and from the proof of Theorem 1.1 in [10] (implication (d) $\Rightarrow$ (a), p. 61), it follows that the respective isometry constructed in [10] is also a lattice homomorphism so that (e) $\Rightarrow$ (a).

LEMMA 2.2. *The following conditions are satisfied for* $E_\phi(\mu)$.

(a) $\|h_n\|_\phi \uparrow 1$ *implies* $I_\phi(h_n) \to 1$, *whenever* $0 \leqslant |h_n| \leqslant f$ *and* $f \in E_\phi(\mu)$.

(b) $E_\phi(\mu)$ *does not contain an isometric copy of* l_∞.

In particular in $E_\phi(\mu)$, *if* $\|f\|_\phi = 1$ *then* $I_\phi(f) = 1$.

Proof. Let us first prove the last assertion. Clearly $I_\phi(\cdot)$ is finite and convex on $E_\phi(\mu)$. Since $I_\phi(h) \leqslant \|h\|_\phi \leqslant 1$ it must be continuous on $E_\phi(\mu)$. Hence the desired property follows.

Applying the same approach as in [10, Lemma 1.5], we obtain with $\alpha_n = 1/\|h_n\|_\phi$ $(\alpha_n \downarrow 1)$ that $1 = I_\phi(\alpha_n h_n) \leqslant (\alpha_n - 1) I_\phi(2h_n) + (2 - \alpha_n) * I_\phi(h_n) \leqslant (\alpha_n - 1) I_\phi(2f) + (2 - \alpha_n) I_\phi(h_n)$. Since $f \in E_\phi(\mu)$, $I_\phi(2f) < +\infty$. Letting $n \to +\infty$ it is seen that it cannot be that $I_\phi(h_{n_k}) \leqslant \alpha < 1$ for any subsequence (n_k). Hence (a) follows.

Clearly $E_\phi(\mu)$ has order continuous norm so by virtue of a general result (see [12, Chap. X, Sect. 4.1]) the condition (b) is satisfied.

LEMMA 2.3. *Let* $\phi > 0$. *Assume that* $f \geqslant g_n \geqslant 0$, $\|f\|_\phi \leqslant 1$, *where either* $f, g_n \in E_\phi(\mu)$, *or* $f, g_n \in L_\phi(\mu)$ *and* $\phi \in \Delta_2$. *Then* $I_\phi(g_n) \to 0$ *implies that* $\|g_n\|_\phi \to 0$.

Proof. Assume to the contrary, that $\alpha_n = \|g_n\|_\phi \geqslant \varepsilon$ for some $\varepsilon > 0$, where without loss of generality $n \in \mathbb{N}$. Applying Lemma 2.2 we obtain that $1 = I_\phi(g_n/\alpha_n) \leqslant (1/\alpha_n - 1) I_\phi(2g_n) + (2 - 1/\alpha_n) I_\phi(g_n)$ with $I_\phi(g_n) \to 0$ and α_n bounded. By the assumption $g_{n_k} \to 0$ a.e. on T with $0 \leqslant g_{n_k} \leqslant f$. If $f \in E_\phi(\mu)$ then $I_\phi(2f) < +\infty$. Also, if $f \in L_\phi(\mu)$ from the assumption $\phi \in \Delta_2$ we have $I_\phi(2f) < +\infty$. Hence, applying the Lebesgue theorem it follows that

$I_\phi(2g_{n_k}) \to 0$. Now, by virtue of the inequality above, we get a contradiction, so the proof is completed (cf. [13]).

Since $\phi \in \Delta_2$ implies $\phi < +\infty$, from Propositions 2.1 and 1.5 it follows

COROLLARY 2.4. *If $L_\phi(\mu)$ is a UM or an STM space then*:

(a) $0 < \phi < +\infty$.

(b) $\phi \in \Delta_2$.

Remark. If ϕ does not depend on $t \in T$ then (b) $\Rightarrow$ (a).

We begin with characterizations of STM and UM properties of $L_\phi(\mu)$ and $E_\phi(\mu)$ in terms of the respective monotonicity properties of $I_\phi(\cdot)$.

PROPOSITION 2.5. *The following pairs ((a), (b)) of statements are equivalent.*

(a) *$L_\phi(\mu)$ is an STM space ($L_\phi(\mu)$ is a UM space).*

(b) (i) *$I_\phi(\cdot)$ is STM modular (resp. $I_\phi(\cdot)$ is UM modular).*

(ii) *$f \in L_\phi(\mu)$, $\|f\|_\phi = 1$ imply that $I_\phi(f) = 1$ (resp. $f_n \in L_\phi(\mu)$, $\|f_n\|_\phi \nearrow 1$ imply that $I_\phi(f_n) \to 1$).*

In view of Proposition 2.1 and the definitions, the proof is evident.

PROPOSITION 2.6. *The following pairs ((a), (b)) of statements are equivalent for the space $E_\phi(\mu)$.*

(a) *$E_\phi(\mu)$ is an STM space ($E_\phi(\mu)$ is an LUM space).*

(b) *$I_\phi(\cdot)$ is STM on $E_\phi(\mu)$ modular (resp. $I_\phi(\cdot)$ is LUM on $E_\phi(\mu)$ modular).*

Proof. To prove (a) $\Rightarrow$ (b) we use the respective definitions and that $\|h\|_\phi \leqslant \xi < 1$ implies $I_\phi(h) \leqslant \xi$. To prove (b) $\Rightarrow$ (a) we apply also Lemma 2.2. Let us point out that $I_\phi(\cdot)$ is LUM on $E_\phi(\mu)$ if for each $f \geqslant 0$ in $E_\phi(\mu)$ with $\|f\|_\phi = 1$ given $\varepsilon > 0$ there exists $\delta > 0$ such that $I_\phi(f - g) < 1 - \delta$ whenever $f \geqslant g \geqslant 0$ and $\|g\|_\phi \geqslant \varepsilon$.

Now, we can prove our main results of the paper.

THEOREM 2.7. *For μ nonatomic the following statements are equivalent.*

(a) *$L_\phi(\mu)$ is an STM space.*

(b) *$L_\phi(\mu)$ is a UM space.*

(c) (i) $\phi > 0$.

(ii) $\phi \in \Delta_2$ *(equivalently, $\|\cdot\|_\phi$ is order continuous).*

Proof. Implication (b) $\Rightarrow$ (a) is clear and (a) $\Rightarrow$ (c) by Corollary 2.4 and Proposition 2.5.

To get (c) $\Rightarrow$ (b) we will prove that the respective conditions (b)(i)–(ii) from Proposition 2.5 are satisfied. First, in view of Proposition 2.1 the condition (b)(ii) follows. Next, in view of Proposition 1.4, the modular $I_\phi(\cdot)$ is always ϕ-UM; i.e., for each $\varepsilon_1 > 0$ there exists $\delta_1(\varepsilon_1) > 0$ such that $I_\phi(f-g) \leqslant 1 - \delta_1(\varepsilon_1)$ whenever $I_\phi(g) \geqslant \varepsilon_1$, $f \geqslant g \geqslant 0$, $I_\phi(f) = 1$. To show the UM of $I_\phi(\cdot)$ it suffices to prove that $I_\phi(g) \geqslant \varepsilon_1$ and $I_\phi(f) = 1$ can be replaced by $\|g\|_\phi \geqslant \varepsilon_1$ and $\|f\|_\phi = 1$, respectively.

In the last case Proposition 2.1 (condition (b)) can be applied. In the first one Proposition 2.2 applies, so that for every $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that $\|g\|_\phi \geqslant \varepsilon$ implies $I_\phi(g) \geqslant \eta(\varepsilon)$. Hence, with $\varepsilon_1 = \eta(\varepsilon)$, we get $\delta(\varepsilon) = \delta_1(\eta(\varepsilon))$ such that $I_\phi(f-g) \leqslant 1 - \delta(\varepsilon)$ whenever $f \geqslant g \geqslant 0$, $\|f\|_\phi = 1$, and $\|g\|_\phi \geqslant \varepsilon$. Now, from Proposition 2.5 it follows that $L_\phi(\mu)$ is a UM space and the proof is finished.

THEOREM 2.8. *For μ-nonatomic and $\phi < +\infty$ the following are equivalent.*

(a) *$E_\phi(\mu)$ is an STM space.*

(b) *$E_\phi(\mu)$ is an LUM space.*

(c) $\phi > 0$.

Proof. Implication (b) $\Rightarrow$ (a) is clear and (a) $\Rightarrow$ (c) follows from Proposition 1.5. To prove (c) $\Rightarrow$ (b) we first prove that $\phi > 0$ implies that $I_\phi(\cdot)$ is LUM modular. In view of Proposition 1.4, $I_\phi(\cdot)$ is always ϕ-UM and hence ϕ-LUM. From Lemma 2.2 and Lemma 2.3 it follows that $I_\phi(\cdot)$ is actually LUM modular. Applying again Lemma 2.2(a) we get that $E_\phi(\mu)$ is an LUM space as desired.

Remark. In general we cannot expect that $E_\phi(\mu)$ is a UM space if $\phi > 0$ only. In fact from [8, p. 753] it is known that if $\phi \notin \Delta_2$ then $E_\phi(\mu)$ contains an isomorphic copy of c_0. Clearly, in this case $E_\phi(\mu)$ is a proper ideal in $L_\phi(\mu)$. From [12, Chap. X, Sect. 4.4] it follows now that $E_\phi(\mu)$ is not a KB space (in fact B-space). On the other hand, as was already mentioned, each UM Banach lattice is a KB space. Therefore $E_\phi(\mu)$ cannot be a UM space.

3. APPLICATIONS TO BEST APPROXIMATION

Let X be a normed lattice, $f \in X$, and $K \subset X$ a convex subset. Denote, $P_K(f) = \{u \in K\colon \|f-u\| = \inf_{h \in K} \|f-h\|\}$. Recall that μ is nonatomic.

(A) The results below indicate some analogy between the role of strict convexity in the unicity problem in best approximation for normed spaces, and the STM property in such problems for normed lattices. We will deal with dominated best approximation, i.e., best approximation with respect to K under the assumption that $f \geqslant K$ ($K \geqslant f$).

PROPOSITION 3.1. *The following statements are equivalent.*

(a) *X is an STM space.*

(b) *For all $f \in X$ and order intervals $[a, b] \subset X$ satisfying $f \geqslant [a, b]$ ($f \leqslant [a, b]$) there holds* $\operatorname{Card}(P_{[a,b]}(f)) \leqslant 1$.

(c) *For all $f \in X$ and all sublattices $K \subset X$ such that $f \geqslant K$ ($f \leqslant K$) there holds* $\operatorname{Card}(P_K(f)) \leqslant 1$, *i.e., the dominated best approximation with respect to sublattices in X is unique.*

Proof. (a) $\Rightarrow$ (c). If not there exist $u, w \in K$, $u \neq w$, such that $\|f-u\| = \|f-w\| = \inf_{h \in K} \|f-h\|$. Since K is a sublattice $u \vee w \in K$. Since $f \geqslant K$, $0 \leqslant f-u \vee w \leqslant f-u$, so $u \vee w \in P_K(f)$. Since $u \neq w$, we have either $u < u \vee w$ or $v < u \vee w$. In the first case $\|f-u\| = \|f-u \vee w\| = \|f-u-(u \vee w-u)\|$, a contradiction to the STM property of X. The second case runs analogously and therefore (c) follows.

(b) $\Rightarrow$ (a). Proceeding by a contradiction, there exist $f \geqslant g \geqslant 0$ such that $g \neq 0$ and $\|f-g\| = \|f\|$. Define $K = [0, g]$. Then $0, g \in P_K(f)$. In view of (b) we get a contradiction.

(c) $\Rightarrow$ (b). This is clear, since order intervals are sublattices.

From Theorems 2.7 and 2.8 it follows now

PROPOSITION 3.2. *The dominated best approximation in $L_\phi(\mu)$ ($E_\phi(\mu)$ with $\phi < +\infty$) with respect to sublattices is unique if and only if $\phi > 0$ and $\phi \in \Delta_2$ (resp. $\phi > 0$).*

Concerning the existence of best approximation let us point out that in $L_\phi(\mu)$ with $\phi \in \Delta_2$, or $E_\phi(\mu)$ with $\phi < +\infty$, we have $P_{[a,b]}(f) \neq \varnothing$. Indeed, under the assumptions $L_\phi(\mu)$ (resp. $E_\phi(\mu)$) has order continuous norm. Thus, equivalently [12, Chap. X, Sect. 4], all order intervals are weakly compact. Now it suffices to note that the norm is weakly lsc. In fact we have a little more.

PROPOSITION 3.3. *For the Musielak–Orlicz space $L_\phi(\mu)$ the following statements are equivalent.*

(a) $\phi \in \Delta_2$.

(b) *The dominated best approximation with respect to closed sublattices always has a solution.*

Moreover, the condition (b) *still holds true for $E_\phi(\mu)$ whenever $\phi < +\infty$.*

Proof. (a)$\Rightarrow$(b). Let $f \geqslant K$ and $h_n \in K$ be a minimizing sequence: $d = \inf_{h \in K} \|f-h\|_\phi = \lim_{n \to +\infty} \|f-h_n\|_\phi$. Since K is a sublattice $u_n = \bigvee_{k=1}^n h_n \in K$. Moreover $0 \leqslant f - u_n \leqslant f - h_n$ implies that u_n is minimizing sequence. Since there exists $u = \bigvee_n u_n \leqslant f$ we have $0 \leqslant u - u_n \downarrow 0$. Applying that $\phi \in \Delta_2$ is equivalent to the order continuity of any lattice norm, Proposition 2.1, we conclude that $u - u_n$ converges in norm to zero. Since K is norm closed $u \in K$. Also $d = \|f-u\|_\phi = \lim_n \|f-u_n\|_\phi$. This yields that $P_K(f) \neq \varnothing$.

(b)$\Rightarrow$(a). It suffices to apply Proposition 2.1 and the scheme of proof of the implication (ii)$\Rightarrow$(iii) in Theorem 10 in [20]. Namely, assuming that $\phi \notin \Delta_2$, i.e., $\|\cdot\|_\phi$ is not order continuous, there exists a sequence f_n such that $0 \leqslant f_n \downarrow 0$ and $\inf_n \|f_n\|_\phi > 0$. Replacing if necessary f_n by $(1+1/n) f_n$ we obtain $\|f_n\|_\phi > \|f_{n+1}\|_\phi$. Then, for the sublattice $K = \{f_n\}$, $0 \leqslant K$, $P_K(0) = \varnothing$. On the other hand K is (norm) closed. Indeed, let for a moment $\|f_{n_k} - g\|_\phi \to 0$ for $f_{n_k} \in K$ and $g \notin K$. Then $f_{n_k} \to g$ in measure and consequently $g = 0$. Thus $\|f_{n_k}\|_\phi \to 0$ which is impossible. Therefore, in view of the our assumptions, $P_K(0) \neq \varnothing$. This contradiction finishes the proof.

Collecting Propositions 3.2, 3.3, and Theorems 2.7, 2.8 we obtain

THEOREM 3.4. *The following are equivalent for Musielak–Orlicz spaces with the Luxemburg norm.*

(a) *The dominated best approximation in* $L_\phi(\mu)$ (*in* $E_\phi(\mu)$ *with* $\phi < +\infty$) *with respect to closed sublattices exists and is unique.*

(b) $L_\phi(\mu)$ (*resp.* $E_\phi(\mu)$ *with* $\phi < +\infty$) *is an STM space.*

(c) $\phi > 0$ *and* $\phi \in \Delta_2$ (*resp.* $\phi > 0$).

In fact a stronger result than Proposition 3.3 can be stated as a corollary from Theorem 6 in [3].

THEOREM 3.5. *For Musielak–Orlicz spaces the following statements are equivalent.*

(a) $\phi \in \Delta_2$.

(b) *For all closed linear sublattices* $K \subset L_\phi(\mu)$, $P_K(f) \neq \varnothing$ *for all* $f \in L_\phi(\mu)$.

Proof. (b)$\Rightarrow$(a). This follows in view of Proposition 3.3. The implication (a)$\Rightarrow$(b) follows from Theorem 6 in [3], which states that for weakly sequentially complete Banach lattices each closed linear sublattice is proximinal. Recall [12, Chap. X, Sect. 4.4] that weakly sequentially complete Banach lattices coincide with KB lattices (Section 1). In view of Proposition 2.1, $L_\phi(\mu)$ is order continuous. Moreover it is easy to prove (cf. [12, Chap. IV, Sect. 3, Theorem 7]) that $L_\phi(\mu)$ is monotonically complete (B-condition) for sequences. Thus $L_\phi(\mu)$ is a KB space and consequently (b) follows.

(B) Our second aim is to point out an application of the STM property in characterization-type theorems. At first we will deal with ideal Banach function spaces $E(\mu)$. Recall [12, Chap. IV, Sect. 3; 25, pp. 415–421], given a σ-finite positive measure space (T, Σ, μ), $E(\mu)$ is an ideal in the space $M(\mu)$ of all μ-measurable functions (functions equal μ-a.e. on T are identified), if $|h| \leqslant |f|$, $f \in E(\mu)$, $h \in M(\mu)$ imply $h \in E(\mu)$ and is a Banach space under a monotone norm $\|\cdot\|_E$ (i.e., $|h| \leqslant |f|$ implies $\|h\|_E \leqslant \|f\|_E$). Clearly, $E(\mu)$ is a Banach lattice and $L_\phi(\mu)$, $E_\phi(\mu)$ with the Luxemburg norm $\|\cdot\|_\phi$ are spaces of this kind.

THEOREM 3.6. *Let $\|\cdot\|_E$ be order continuous, $K \subset E(\mu)$ a convex subset, and $f \in E(\mu) \backslash K$, $f_0 \in K$. The following statements are equivalent (μ arbitrary, σ-finite).*

(a) $f_0 \in P_K(f)$.

(b) *There exists a μ-measurable function g in the associated space $E'(\mu)$ satisfying $\|g\|_* \; (=\sup_{\|h\|_E \leqslant 1} |\int_T h(t)\, g(t)\, d\mu|) = 1$ and such that*

(i) $\int_T |f(t) - f_0(t)|\, |g(t)|\, d\mu = \|f_0 - f\|_E$,

(ii) $\operatorname{sign}(g(t)) = \operatorname{sign}(f(t) - f_0(t))$, *$\mu$-a.e. on* $\operatorname{Supp}(f - f_0) \cap \operatorname{Supp}(g)$,

(iii) $\int_T (f_0(t) - h(t))\, g(t)\, d\mu \geqslant 0$, *for all $h \in K$.*

Let (a) *be satisfied. If $E(\mu)$ is an STM space then* $\operatorname{Supp}(f - f_0) \subset^\mu \operatorname{Supp}(g)$. *If the associated space $E'(\mu)$ is an STM space under the (associated) norm $\|\cdot\|_*$ then* $\operatorname{Supp}(f - f_0) \supset^\mu \operatorname{Supp}(g)$.

Proof. Recall that $E'(\mu) = \{g \in M(\mu) \colon |\int_T fg\, d\mu| < +\infty$ for all $f \in E(\mu)\}$ and under the order continuity of $\|\cdot\|_E$ each functional $l \in E^*(\mu)$ has the integral representation, i.e., $l(f) = \int_T fg\, d\mu$ for all $f \in E(\mu)$, where $g \in E'(\mu)$ is unique and such that $\|g\|_* = \|l\|$ (the dual norm of l). Moreover $\int_T |f(t)|\, |g(t)|\, d\mu \leqslant \|f\|_\phi \|g\|_*$ (cf. [25, Corollary 106.4 and Sect. 112]).

(a) $\Rightarrow$ (b). From a general characterization theorem [22, Theorem 5.1] there exists a norm-one functional $l \in E^*(\mu)$, now uniquely represented by a function $g \in E'(\mu)$, such that $l(f - f_0) = \|f - f_0\|_E$ and $l(f_0 - h) \geqslant 0$ for all $h \in K$. Thus, $\|g\|_* = 1$ and (iii) follows. Since $E(\mu)$ is an ideal Banach function space $\|g\|_* = \|\,|g|\,\|_*$. Therefore $\|f - f_0\|_E = \int_T (f(t) - f_0(t))\, g(t)\, d\mu \leqslant \int_T |f(t) - f_0(t)|\, |g(t)|\, d\mu \leqslant \|f - f_0\|_E \, \|\,|g|\,\|_*$ imply conditions (i) and (ii).

(b) $\Rightarrow$ (a). In view of the mentioned theorem from [22] the implication is clear.

To prove the remaining part of the theorem let $T_h = \mathrm{Supp}(h) \setminus \mathrm{Supp}(g)$ and $T_g = \mathrm{Supp}(g) \setminus \mathrm{Supp}(h)$, where $h = f - f_0$. Then $\|h\|_E = \int_T h(t)\, g(t)\, d\mu \leqslant \|\, |h|\, 1_{T \setminus T_h} \|_E \leqslant \|h\|_E$. Hence, in view of the STM property for $E(\mu)$, it follows that $\mu(T_h) = 0$. Hence $\mathrm{Supp}(f - f_0) \subset^\mu \mathrm{Supp}(g)$. Starting with the equality $\int_T |h(t)|\, |g(t)|\, d\mu = \|h\|_\phi \|g\|_*$ the last assertion can be proved analogously. Thus, $\|\, |g|\, 1_{T \setminus T_g} \|_* = \|\, |g|\, \|_*$ and hence $\mu(T_g) = 0$. Therefore $\mathrm{Supp}(f - f_0) \supset^\mu \mathrm{Supp}(g)$ as desired.

Remark. It follows that if $E'(\mu)$ is an STM space with the associated norm, the sign of g is fully determined on $\mathrm{Supp}(g)$ by the sign of $f - f_0$. If $E(\mu)$ is an STM space the sign of g is fully determined on $\mathrm{Supp}(f - f_0)$ only.

If $E(\mu)$ is a Musielak–Orlicz space $L_\phi(\mu)$ a more complete result is possible. To avoid an explicit characterization of STM for the associated space $E'(\mu) = L_{\phi^*}(\mu)$ with the Orlicz norm $\|g\|_* = \sup_{\|f\| \leqslant 1} |\int_T f(t)\, g(t)\, d\mu|$ we proceed in different way. We apply another condition in terms of the function ϕ (cf. [17, Lemma 3.1]), ensuring that $\mathrm{Supp}(g) \subset^\mu \mathrm{Supp}(f - f_0)$.

THEOREM 3.7. *Let ϕ satisfy the Δ_2 condition and $K \subset L_\phi(\mu)$ be a convex subset. For $f_0 \in K$ and $f \in L_\phi(\mu) \setminus K$ the following are equivalent.*

(a) $f_0 \in P_K(f)$.

(b) *There exists a function $g_0 \in L_{\phi^*}(\mu)$ satisfying*

(i) $|g_0(t)| \in \partial\phi(|f(t) - f_0(t)| / \|f - f_0\|_\phi, t)$ *μ-a.e. on T.*

(ii) $\mathrm{sign}(g_0(t)) = \mathrm{sign}(f(t) - f_0(t))$ *μ-a.e. on* $\mathrm{Supp}(f - f_0) \cap \mathrm{Supp}(g_0)$.

(iii) $\int_T (f_0(t) - h(t))\, g_0(t)\, d\mu \geqslant 0$ *for all $h \in K$.*

Let (a) *be satisfied. If $\phi > 0$ then* $\mathrm{Supp}(f - f_0) \subset^\mu \mathrm{Supp}(g_0)$. *If ϕ is smooth at zero (i.e., for μ-a.a. $t \in T$ $\inf_{r > 0} \phi(r, t)/r = 0$) then* $\mathrm{Supp}(g_0) \subset^\mu \mathrm{Supp}(f - f_0)$.

Remark. The subdifferential in (i) is taken at the points $|f(t) - f_0(t)| / \|f - f_0\|_\phi$ for $t \in T$. Recall that $\beta \in \partial\phi(\alpha, t)$ if and only if $\alpha\beta = \phi(\alpha, t) + \phi^*(\beta, t)$, where $\alpha, \beta \geqslant 0$ and ϕ^* denotes the Young conjugate to ϕ [11].

Proof. An outline of the proof will be given only. Since $\phi \in \Delta_2$, i.e., the norm is order continuous, we have $E'(\mu) = L_{\phi^*}(\mu)$ with the dual norm $\|g\|_* = \sup_{\|h\|_E \leqslant 1} |\int_T h(t)\, g(t)\, d\mu|$. To get this one can proceed as in [25, Sects. 132 and 133] with Musielak–Orlicz spaces [24] instead of Orlicz spaces.

Now the condition (b)(i) from Theorem 3.4 means that the functional l defined by $l(h)=\int_T h(t)\,|g(t)|\,d\mu$, $h\in L_\phi(\mu)$, satisfies $l\in\partial\,\|u\|_\phi$ with $u=|f-f_0|/\|f-f_0\|_\phi$. Thanks to $\phi\in\Delta_2$ the modular I_ϕ is continuous on $L_\phi(\mu)$. Proceeding as in [17, Lemma 3.1], we obtain that $l=k/\|k\|$ with $k\in\partial I_\phi(u)$, i.e., $k(u)=I_\phi(u)+(I_\phi)^*\,(k)$. Clearly, k has still the integral representation by $g_1\in L_\phi(\mu)$, with $\|g_1\|_*=\|k\|$. Therefore $|g(t)|=g_1(t)/\|k\|$. Rockafellar's representation theorem for convex integral functionals [15, 16] yields now that $(I_\phi)^*\,(k)=I_{\phi^*}(g_1)$. Consequently in the Young inequality $\phi(u(t),t)+\phi^*(g_1(t),t)\geqslant u(t)\,|g_1(t)|$ we have equality which means that $g_1(t)\in\partial\phi(u(t),t)$ for μ-a.e. $t\in T$. Thus (b)(i) follows with $g_0\in L_{\phi^*}(\mu)$ such that $|g_0(t)|=g_1(t)$ and $\operatorname{sign}(g_0(t))=\operatorname{sign}(g(t))$. By the way (b)(ii) follows from Theorem 3.4.

It remains to prove the second part of the theorem. Let $\phi>0$, $\phi\in\Delta_2$. From Theorem 2.7 it follows that $L_\phi(\mu)$ is an STM space, so by Theorem 3.4, $\operatorname{Supp}(f-f_0)\subset^\mu\operatorname{Supp}(g)$. Next, let ϕ be smooth at zero. From (b)(i), $|h(t)|\,|g_0(t)|=\phi(|h(t)|,t)+\phi^*(|g_0(t)|,t)$ where $h=f-f_0$ and $t\in T$. Therefore $\phi^*(|g_0(t)|,t)=0$ on $S=\operatorname{Supp}(g_0)\backslash\operatorname{Supp}(f-f_0)$. Since $\phi^*(|g_0(t)|,t)=\sup_{r>0}(r\,|g_0(t)|-\phi(r,t))$ we conclude that $|g_0(t)|\leqslant\phi(r,t)/r$ for $t\in S$ and $r>0$. Therefore $\operatorname{Supp}(g_0)\subset^\mu\operatorname{Supp}(f-f_0)$.

Remarks. 1. The same theorem is still true with $E_\phi(\mu)$ $(\phi<+\infty)$ instead of $L_\phi(\mu)$, without the Δ_2-condition. The proof runs analogously.

2. The same theorem can be derived from Theorem 3.6 and the main theorem from [19] for μ purely atomic.

EXAMPLES. If $\phi(r,t)=r$ $(r\geqslant 0)$ then $L_\phi(\mu)$ reduces to $L_1(\mu)$. From Theorem 3.7 it follows immediately that $|g_0(t)|=1$ and $\operatorname{sign}(g_0(t))=\operatorname{sign}(f(t)-f_0(t))$ μ-a.e. on $\operatorname{Supp}(f-f_0)$. However, on the remaining part of T, $|g_0(t)|\in[0,1]$ is any measurable selector (such that $g\in L_{\phi^*}(\mu)$) where $\operatorname{sign}(g(t))$ can vary arbitrarily since $\phi(r,t)$ is not smooth at zero.

If $\phi(r,t)=t^p$ $(r\geqslant 0,\ 1<p<+\infty)$ the well known form of the function g follows. Namely, in this case for μ-a.e. on $T\,|g(t)|=|f(t)-f_0(t)|^{p-1}/\|f-f_0\|_p$ and $\operatorname{sign}(g(t))=\operatorname{sign}(f(t)-f_0(t))$ [22]. Let us point out that the spaces $L_p(\mu)$ under consideration are STM spaces (in fact UM spaces).

(C) Let (T,Σ,μ) be a probability measure space and $E(\mu)$ an ideal Banach function space defined in (B) satisfying $L_\infty(\mu)\subset E(\mu)\subset L_1(\mu)$. In Proposition 4 from [3] it is proved given a sub-σ-algebra $\Sigma_0\subset\Sigma$ and the corresponding subspace $E(\mu\,|\,\Sigma_0)\subset E(\mu)$ that every minimizing sequence $(f_n)\in E(\mu\,|\,\Sigma_0)$, such that $\|f-f_n\|_E\to\inf(\|f-h\|_E\colon h\in E(\mu\,|\,\Sigma_0)\}$, is $E(\mu)$-equi-integrable whenever $E(\mu)$ is a UM-space. Moreover, in the case under consideration the set $P_{E(\mu|\Sigma_0)}(f)$ of all best approximations for f with respect to $E(\mu\,|\,\Sigma_0)$ is (convex) weakly compact. Hence on the basis of Theorem 2.7 we get a more complete result for Musielak–Orlicz spaces.

THEOREM 3.8. *Let $L_\phi(\mu)$ be a Musielak–Orlicz space satisfying $L_\infty(\mu) \subset E(\mu) \subset L_1(\mu)$ with $\phi \in \Delta_2$, $\phi > 0$, and μ nonatomic. Then*

(a) *If $\|f - f_n\|_E \to \inf\{\|f-h\|_E: h \in E(\mu \mid \Sigma_0)\}$, then (f_n) is $L_\phi(\mu)$-equi-integrable.*

(b) *The set $P_{E(\mu \mid \Sigma_0)}(f)$ is (convex) weakly compact.*

Referring to [3] some further results for Musielak–Orlicz spaces also follow.

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